On the Steady Flow of a Non-Newtonian Fluid in Cylinder Ducts

ROBERT S. SCHECHTER

The University of Texas, Austin, Texas

The flow of a power law fluid in cylindrical ducts is considered, with a variational principle evolved from minimum entropy considerations used. This principle is applied to flow in rectangular ducts and between two flat plates with the method of Ritz and Galerkin to determine the velocity profiles used. These velocity profiles are used to determine the relationship between the friction factor and the Reynolds number.

Variational principles have been used with considerable success in dealing with mechanical systems. Recently Rosen (*1*) has described a variational approach to magnetohydrodynamics which is based on the early work of Onsager (*2*) in irreversible thermodynamics. The Hemholtz (*3*) principle is applicable to Newtonian fluids in steady motion and for which the velocity is specified on every point of the boundary. Serrin (*4*) has discussed the Herivel-Lin formulation which is, qualitatively at least, based on Hamilton's principle. It appears that this latter formulation is not in a form which is usable for computational purposes. The problem of a solid sphere moving steadily through a non-Newtonian fluid has been formulated for the so-called "power law fluid" (*5, 6*) by Tomita (*7*) using a variational approach. The basis for Tomita's argument was essentially Prigogine's (*8*) principle of minimum entropy production. However Bird (*9*) in devising a variational principle for a rather general class of non-Newtonian fluids has concluded that the principle of minimum entropy production is valid only for Newtonian flow and for the flow of a power law fluid. Bird's principle reduces to that of Hemholtz for a Newtonian fluid and to that of Tomita for a power law fluid. It should be pointed out that Bird's principle, while applicable to a very general fluid, is extremely restricted in practical use because it is only valid for the steady, incompressible flow of a non-Newtonian fluid in those cases in which the inertia terms are negligible and for systems in which the velocity is specified at every point on a closed boundary.

The work presented in this paper is concerned with the steady flow of an incompressible power law fluid through a cylindrical duct. Such a configuration is shown in Figure 1*a*. If it is assumed that the velocity components are such that

$$V_1 = V_2 = 0,\ V_3 = V_3(x_1, x_2) \neq 0$$

and that the stress rate of strain relationship is given by

$$\tau_{ij} = K\left\{\frac{1}{2}\sum_r\sum_s \Delta_{rs}\Delta_{sr}\right\}^{\frac{n-1}{2}}\Delta_{ij}$$

$$i, j, r, s, = 1, 2, 3 \qquad (1)$$

then $V_3(x_1, x_2)$ will be such that the integral

$$I = \int_A\int \left\{K\left[\left(\frac{\partial V_3}{\partial x_1}\right)^2 + \left(\frac{\partial V_3}{\partial x_2}\right)^2\right]^{\frac{n+1}{2}} - \lambda\frac{dp}{dx_3}V_3\right\}dA \qquad (2)$$

is a minimum.

$$\Delta_{rs} = \frac{\partial V_r}{\partial x_s} + \frac{\partial V_s}{\partial x_r} \qquad (3)$$

is the strain-rate tensor.

The empirical formulation of the stress rate of strain relationship expressed by Equation (1) has yet to be verified experimentally except for simple rectilinear motions. To check this relationship for more complex situations experimental results must be compared with theoretical values obtained by solving the equation of motion. Velocity distributions for flow of a power law fluid in cylindrical ducts evolved with the variational approach suggested in this paper should serve as a basis for assessing the validity of the power law model when experimental data become available.

The validity of the variational formulation expressed by Equation (2) is easily demonstrated by applying the Euler-Lagrange (*10*) equation to the integrand and observing that the resultant expression is the equation of motion provided the Lagrangian multiplier is chosen as

$$\lambda = -(n+1) \qquad (4)$$

Equation (2) is simply an expression of the principle of minimum entropy production as applied to the steady flow of a power law fluid in a cylindrical duct. In this case the rate of entropy production is minimized subject to the restriction that the rate of dissipation of mechanical energy per foot of tube is constant. It should be emphasized however that this argument will not lead to the correct variational principle for all possible types of non-Newtonian fluids. Moreover a general variational principle valid for all possible fluids in all possible flow configurations has not been proposed, although such a formulation would be valuable in performing computations as well as formulating equations of motion. For Newtonian fluids ($n = 1$) Equation (2) reduces to the equation proposed by Sparrow and Siegel (*11*).

GENERAL CONSIDERATIONS

The problem is to find the velocity distribution which minimizes the integral I. To reduce the expression to dimensionless form the following variables are defined:

$$V' = V_3/V_{avg}$$

$$\Phi = \frac{1}{2}(n+1)\,N_{Re}f$$

$$A' = A/L^2$$

$$I' = \frac{I}{K V^2_{avg}}\left(\frac{L}{V_{avg}}\right)^{n-1}$$

$$x'_1 = x_1/L$$

$$x'_2 = x_2/L$$

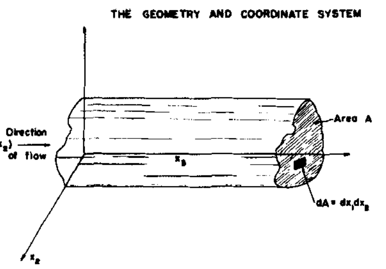

Fig. 1a. General case.

$$f = -\frac{dp}{dx_3}\left(\frac{2L}{\rho\, V^2_{\text{avg}}}\right)$$

$$V_{\text{avg}} = \frac{1}{A}\int_A\int V_3(x_1, x_2)\, dA$$

$$N_{Re} = \frac{\rho\, V^2_{\text{avg}}}{K}\left(\frac{L}{V_{\text{avg}}}\right)^n$$

Substituting these definitions into Equation (2) one obtains

$$I' = \int_{A'}\int \left\{\left[\left(\frac{\partial V'}{\partial x_1'}\right)^2 + \left(\frac{\partial V'}{\partial x_2'}\right)^2\right]^{\frac{n+1}{2}} - \Phi V'\right\} dA' \tag{5}$$

The velocity profiles can be approximated for a particular geometry and fluid (n specified) by the method of Ritz and Galerkin (*12*). The technique envolves choosing a functional expression for the velocity distribution which includes a number of arbitrary constants, a_1, a_2, $\ldots a_i$. Kantorovich and Krylov (*12*) have shown that if the assumed function (in this case the velocity distribution) satisfies the boundary conditions for all values of the arbitrary constants and if the arbitrary function is complete, convergence to the true solution is assured. In actual practice only a few terms of the complete function are required to obtain a good approximation to the exact solution. The accuracy of the truncated series will be examined in subsequent sections. After a velocity distribution which satisfies the conditions specified above is selected, the constants $a_1, a_2, \ldots, a_i$ are fixed so as to minimize I'. Thus, if

$$V' = V'(x_1', x_2', a_1, a_2, \ldots, a_i) \tag{6}$$

TABLE 1. CONSTANTS PERTINENT TO FLOW BETWEEN FLAT PLATES

n	a_1	a_2	$N_{Re}f$
1.0	6.0	0.00	24.00
0.75	5.770	4.598	16.62
0.50	5.3673	12.683	11.33

$$N_{Re}f = 4\left\{\frac{2(2n+1)}{n}\right\}^n$$

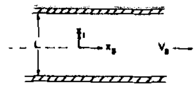

Fig. 1b. Flow between two flat plates.

then the a_j's are to be chosen such that

$$\frac{\partial I'}{\partial a_j} = 0 \quad j = 1, 2, \ldots, i \tag{7}$$

It is interesting to note that the parameter Φ is not known a priori, but Φ is unique for each geometry and value of n since

$$A' = \int_{A'}\int V' dA' \tag{8}$$

in view of the definition of V'.

Thus Φ must be selected so that Equation (8) is satisfied, and as a consequence

$$N_{Re}f = \text{constant} \tag{9}$$

for laminar flow in a cylindrical tube. One of the purposes of the arguments to follow is to establish the value of this constant for the flow of a power law fluid through a rectangular duct.

The computations are divided into two sections, as the special case of two infinite flat plates is considered separately from other rectangular cross sections.

FLOW BETWEEN TWO INFINITE FLAT PLATES

The flow between two flat plates is of interest since the equation of motion can be integrated and a comparison of the exact velocity profile and the approximate results obtained by the Ritz-Galerkin method is possible.

For steady flow between two infinite flat plates Equation (5) reduces to

$$I' = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left\{\left(\frac{\partial V'}{\partial x_1'}\right)^{n+1} - \Phi\, V'\right\} dx_1' \tag{10}$$

if the coordinate system is positioned as shown in Figure 1*b* and if the characteristic length is selected as the distance separating the two plates. To apply the method of Ritz and Galerkin the velocity was assumed to be of the form

TABLE 2. VALUES OF CONSTANTS IN EQUATION (13)

i	α_i	β_i
1	π	π/E
2	3π	π/E
3	π	$3\pi/E$
4	3π	$3\pi/E$
5	5π	π/E
6	π	$5\pi/E$

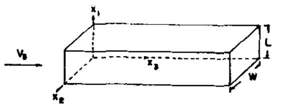

Fig. 1c. Flow in a rectangular duct.

$$V' = \left(\frac{1}{4} - x_1'^2\right)(a_1 + a_2 x_1'^2) \tag{11}$$

This choice has the advantage of reducing to the exact solution for Newtonian fluids. The function also satisfies the boundary conditions for all values of the arbitrary constants. However the function given by Equation (11) is not complete and hence does not satisfy the second of the conditions set forth by Kantorovich and Krylov. It should be noted that an infinite power series in x_1' is a complete set and Equation (11) is a truncated power series. There does not appear to be a simple method of estimating the accuracy of the result when a truncated function is used. A possible procedure might entail adding additional terms to Equation (11) and comparing the higher-order approximation to the two-constant equation. If the difference in the velocity distribution as determined from the two approximations is not significant, it might be concluded that the approximation is satisfactory. In this case the approximation can be compared directly with the theoretical equation, and such a procedure is not required. The coefficients of the odd powers vanish because of symmetry. The two arbitrary constants are to be determined so as to minimize I'. Since the integration required by Equation (10) is not possible for all values of n in the interval (0, 1), a numerical integration with a Gaussian quadrature (*13*) was coupled with an iterative procedure based on the modified method of steepest descent as proposed by Marquardt (*14*) in order to determine the values of the a_j's which satisfy Equation (7). These values are given in Table 1 as a function of the pa-

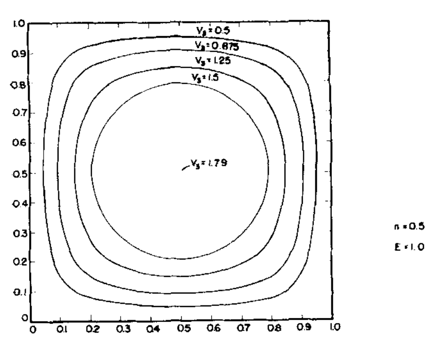

Fig. 2. Typical velocity profile, flow of power-law fluid in square duct.

rameter n. The approximate velocity profile can be compared with the exact velocity distribution as computed from the following expression which is a solution to the equation of motion:

$$V' = \left(\frac{2n+1}{n+1}\right)\{1-(2|x_1'|)^{\frac{n+1}{n}}\} \quad (12)$$

Since Equation (11) with $a_1 = 6.00$ and $a_2 = 0.0$ is exact for Newtonian fluids, it is reasonable to suspect that the approximate velocity should become less accurate as n is decreased from 1. Thus a comparison for n = 0.5 should be a severe test of the method. The computed results are less than 1% different than the true velocities except for the region near the wall where the velocity becomes small. The success of this method of approximation is exciting, as it appears that a number of chemical engineering problems can be solved efficiently by a variational approach.

FLOW IN RECTANGULAR DUCTS

The velocity distribution for flow in rectangular ducts was assumed to be of the form

$$V' = \sum_{i=1}^{6} a_i \sin \alpha_i x_1' \sin \beta_i x_2' \quad (13)$$

where α_i and β_i are given in Table 2, the coordinate system is fixed as shown in Figure 1c, the characteristic dimension is taken as the longest side of the rectangle, and the aspect ratio is the ratio of the shorter side to the longer side. This functional form was selected because it satisfies the boundary conditions for all values of the arbitrary constants and it is a truncation of a complete set of functions. Furthermore the exact solution for laminar flow in a rectangular duct can be written in the form of a double Fourier sine series. This latter solution can be written in the form

$$V' = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(2n-1)\pi x_1' \sin\frac{(2m-1)\ (\pi x_2')}{E} \quad (14)$$

where

$$A_{mn} = \frac{2}{(2m-1)(2n-1)\pi^4}\left\{\frac{\Phi}{(2n-1)^2+\frac{1}{E^2}(2m-1)^2}\right\}$$

Table 3. Computed Results for Flow in Rectangular Duct

n	E	A_1	A_2	A_3	A_4	A_5	A_6	Ref.
1.0	1.0	2.346	0.156	0.156	0.0289	0.0360	0.0360	57.08
0.75	1.0	2.313	0.205	0.205	0.0007	0.0434	0.0434	36.34
0.50	1.0	2.263	0.278	0.278	−0.0285	0.0555	0.0555	23.02
1.0	0.75	2.341	0.204	0.119	0.0256	0.0498	0.0303	77.85
0.75	0.75	2.310	0.235	0.180	0.0001	0.0568	0.0364	47.51
0.50	0.75	2.263	0.286	0.267	−0.0277	0.0644	0.0505	28.85
1.0	0.50	2.311	0.296	0.104	0.0285	0.0795	0.0303	140.4
0.75	0.50	2.288	0.299	0.174	0.0120	0.0811	0.0364	105.0
0.5	0.50	2.249	0.312	0.274	−0.0101	0.0789	0.0501	44.33
1.0	0.25	2.227	0.503	0.0867	0.0274	0.184	0.0189	459.4
0.75	0.25	2.221	0.459	0.160	0.0312	0.160	0.0210	220.9
0.5	0.25	2.205	0.407	0.270	0.0257	0.131	0.0364	105.0

Furthermore it is a simple task to show that the six constants which are to be computed by the Ritz-Galerkin method are identical to the corresponding Fourier coefficients when the fluid is Newtonian; that is

$$a_1 = A_{11},\ a_2 = A_{21},\ a_3 = A_{12},\ a_4 = A_{22},$$
$$a_5 = A_{31},\ a_6 = A_{13} \quad (15)$$

for Newtonian fluids. These relationships are useful in checking the computational procedure.

Again calculations were performed numerically with Simpson's rule in a form suitable for double integration (13), and iterations were based on corrections provided for by the method of steepest descent. The computed results are shown in Table 3. The a_i's are tabulated as a function of both the aspect ratio and n, and in each case the product $N_{Re}f$ is also given. A typical velocity profile is shown in Figure 2.

The accuracy of the computed results can be assessed if the fluid is Newtonian. In this case the value of Φ, which is obtained by integrating the velocity V' over the surface of the rectangle and applying Equation (8), is given by the following equation:

$$\Phi = \frac{\pi^6}{32}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2(2n-1)^2\left[(2n-1)^2+\frac{1}{E^2}(2m-1)^2\right]} \quad (16)$$

The Φ's for laminar flow have been tabulated by Jakob (*15*) (although in a somewhat different form) for aspect ratios of 1.0, 0.5, and 0.25. These values of Φ have been used in computing the Fourier coefficients as defined in Equation (14). By comparing these values of the Fourier coefficients with those obtained from the variational approach an estimate of the magnitude of the errors introduced by both the numerical approximations and the truncation of the Fourier series is obtained. The difference between these coefficients was never greater than 1% and usually much less than 1%. Again the results appear to be accurate enough for most engineering needs.

The results presented in this section should be useful in determining the pressure drop associated with the flow of a power law fluid in a rectangular duct, and also an approximate velocity is given in functional form which can now be applied to the solution of other problems such as the heat transfer problem.

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NOTATION

- A = cross-sectional area
- a_i = constants selected to minimize an integral
- A_{mn} = Fourier coefficients
- E = aspect ratio
- f = friction factor
- I = variational integral
- K = parameter in power law stress rate of strain relationship
- L = characteristic dimension
- n = exponent in power law stress rate of strain relationship
- N_{Re} = Reynolds number
- p = pressure
- V_1, V_2, V_3 = components of velocity in x_1, x_2, x_3 directions, respectively
- V_{avg} = average velocity
- x_1, x_2, x_3 = Cartession coordinates
- Δ_{rs} = component of strain-rate tensor
- λ = Lagrangian multiplier
- ρ = fluid density
- τ_{rs} = component of stress tensor

Φ = dimensionless group related to product of Reynolds number and friction factor

Primed quantities are dimensionless.

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Computer Calculation of Binary-Drop Evaporation

NORMAN E. RAWSON, JOHN S. CHINN, and WILLIAM F. STEVENS

Northwestern University, Evanston, Illinois

A mathematical model has been developed relating the velocity, temperature, composition, radius, and position of an evaporating drop falling through a heated environment. Under the assumption of rapid mixing within the drop a digital computer program has been written and solved for several initial conditions to give the aforementioned variables as a function of time. Details of the computer program are presented, with emphasis on the analysis of and approach to the problem. The computer results are summarized, and their relation to the general problem of spray vaporization is indicated.

A knowledge of the mechanism of vaporization of liquid drops would add to the over-all understanding of many important chemical and physical processes, such as spray drying, combustion of liquid fuels, and air humidification. To date investigations in these areas have been largely of an experimental nature, involving only single-component drops (*6, 7*). Verification of an assumed mechanism for a binary-drop vaporization by calculation is difficult because a binary drop is one made up of two distinct components of different volatilities, such that the drop composition changes during evaporation, along with the drop temperature, velocity, direction, and size.

With several simplifying assumptions, summarized in detail by Chinn (*2*), Culverwell (*3*), and Rawson (*9*), two mathematical models have been developed to represent the evaporation of binary drops. The first is called the *rapid mixing* model. It assumes that mass and temperature gradients within the drop are instantaneously removed by convection currents, so that the drop temperature and composition can be represented by average values. This model is approached in larger drops where rapid convection currents are known to exist. The second model assumes no mixing and would be most likely to occur in smaller drops with negligible convection currents. This model requires solution of partial differential equations to determine the temperature and concentration profiles within the drop. The present paper presents the details of a digital computer solution for the rapid mixing model. Computer solution of the no-mixing model is more complicated and is presently under investigation.

Norman E. Rawson is with The Pure Oil Company, Crystal Lake, Illinois. John S. Chinn is at Wyandotte Chemicals Corporation, Detroit, Michigan.

PROBLEM ANALYSIS

A system of 37 wt. % tetrachloroethylene, with the remainder orthodichlorobenzene, was chosen for study, since these materials have volatilities similar to those of common jet fuels. Histories of 50- to 400-μ drops were calculated as they evaporated in air at a constant temperature of 600°F. The calculation was begun after the drop was ejected from a nozzle. The initial drop temperature was assumed to be 70°F., the initial velocity 49.5 ft./sec., and the initial trajectory 58.5 deg. to the horizontal.

Calculation of the drop history can be divided into three distinct parts: velocity and trajectory, mass transfer, and heat transfer. The velocity and trajectory equations are derived by applying a force balance to the drop. The resultant force acting on the drop is composed of acceleration due to gravity, the buoyancy effect of the surrounding media, and the frictional resistance of the air.

For the horizontal component of motion

$$m_d\, dv_h/d\theta = -F' \cos \phi \qquad (1)$$

For the vertical component of motion

$$m_d\, dv_v/d\theta = g(\rho_d V_d - \rho_a V_a) - F' \sin \phi \qquad (2)$$

For spheres the frictional force from air resistance is

$$F' = \frac{1}{2}\,(f\, A_x\, \rho_a\, v^2) \qquad (3)$$

Brown (*1*) presents a graphical representation of f as a function of the Reynolds number, the curve of which has been approximated analytically by Chinn (*2*) and has been used in the